1 Myerson-Satterthwaite theorem

Let a risk neutral seller have a single object for sale and a risk neutral buyer a unit demand. The seller's type c is distributed on $[\underline{c}, \overline{c}]$ and the buyer's type v on $[\underline{v}, \overline{v}]$. Let the joint distribution be independent and let the marginal distributions over $[\underline{c}, \overline{c}]$ and $[\underline{v}, \overline{v}]$ have strictly positive density. In a bargaining game between the seller and buyer each party observes his own type. The seller chooses an action $\sigma \in S$ and the buyer chooses $\beta \in B$. A pure strategy of the seller is $\phi_s : [\underline{c}, \overline{c}] \to S$ and a pure strategy of the buyer is $\phi_b : [\underline{v}, \overline{v}] \to B$. An action profile (σ, β) results in an outcome via outcome functions $\pi : S \times B \to [0, 1]$, $\tau_s : S \times B \to R$ and $\tau_b : S \times B \to R$. Here π is the probability that the buyer gets the object, τ_s is the payment received by the seller and τ_b the payment of the buyer.

Assume that a type profile results in outcome (q, t_s, t_b). The the payoff to the seller is

$$(1 - q)c + t_s \tag{1}$$

and the payoff to the buyer is

$$qv - t_b \tag{2}$$

Because of risk neutrality the payment can be dealt with as above and it can be thought of as expected payment.

If (ϕ_s, ϕ_b) is a Nash equilibrium ϕ_s maximises the expected payoff for each $c \in [\underline{c}, \overline{c}]$ where the expectation is taken over the buyer's action which depends on the buyer's unknown (to the seller) type. Fix (ϕ_s, ϕ_b) and let us define some more notation. Given c and $\phi_s(c) = \sigma(c)$ define the seller's expected probability of trading and expected payment

$$Q_s(c) = E_v \left(\pi \left(\sigma(c), \phi_b(v)\right)\right) \tag{3}$$

and

$$T_s(c) = E_v \left(\tau \left(\sigma(c), \phi_b(v)\right)\right) \tag{4}$$

Similarly we can define the corresponding quantities for the buyer

$$Q_b(v) = E_c \left(\pi \left(\phi_s(c), \beta(v)\right)\right) \tag{5}$$

$$T_b(v) = E_c \left(\tau \left(\phi_s(c), \beta(v)\right)\right) \tag{6}$$

given the buyer's type v and $\phi_b(v) = \beta(v)$. In the Nash equilibrium the seller's expected payoff is

$$U_s(c) = \left(1 - Q_s(c)\right) c + T_s(c) \tag{7}$$

and the buyer's expected payoff is

$$U_b(v) = Q_b(v)v - T_b(v) \tag{8}$$

Definition 1 *A Nash equilibrium of a bargaining game is ex post efficient iff, for any* (c, v) *and the resulting* (q, t_s, t_b) $q = 1$ *if* $c < v$ *and* $q = 0$ *if* $c > v$.

Definition 2 *A Nash equilibrium of a bargaining game satisfies budget balance iff, for any* (c, v) *and the resulting* (q, t_s, t_b), $t_s = t_b$.

Definition 3 *A Nash equilibrium of a bargaining game is individually rational iff, for any* (c, v), $U_s(c) \geq c$ *and* $U_b(v) \geq 0$.

Theorem 4 *(MS) If* $\underline{v} \leq \overline{c}$ *and* $\underline{c} \leq \overline{v}$ *then there does not exist any Nash equilibrium of any bargaining game that is ex post efficient, satisfies budget balance, and is individually rational.*

Proof. Take any bargaining game and fix its Nash equilibrium (ϕ_s, ϕ_b).

1. First we note some auxiliary results. No deviation in a Nash equilibrium is profitable, and in particular a seller of type c should not profit by playing $\phi_s(c^*) = \sigma(c^*)$ which would yield payoff

$$U_s(c^*) = (1 - Q_s(c^*))\, c + T_s(c^*) \tag{9}$$

Thus the following has to hold

$$U_s(c) = (1 - Q_s(c))\, c + T_s(c) \geq (1 - Q_s(c^*))\, c + T_s(c^*) \tag{10}$$

and also the converse

$$U_s(c^*) = (1 - Q_s(c^*))\, c^* + T_s(c^*) \geq (1 - Q_s(c))\, c^* + T_s(c) \tag{11}$$

(10) and (11) imply that if $c^* > c$ then $U_s(c^*) \geq U_s(c)$. We can also conclude that

$$U_s(c^*) - U_s(c) \geq (1 - Q_s(c))\, (c^* - c) \tag{12}$$

and similarly

$$U_s(c^*) - U_s(c) \leq (1 - Q_s(c^*))\, (c^* - c) \tag{13}$$

Together (12) and (13) imply that

$$(1 - Q_s(c))\, (c^* - c) \leq U_s(c^*) - U_s(c) \leq (1 - Q_s(c^*))\, (c^* - c) \tag{14}$$

This means that $1 - Q_s$ is weakly increasing and Q_s is weakly decreasing. This means that $1 - Q_s$ is Riemann-integrable, and by the definition of Riemann-integral (14) implies that

$$U_s(\overline{c}) - U_s(c) = \int_c^{\overline{c}} (1 - Q_s(x))\, dx \tag{15}$$

for any $c \in [\underline{c}, \overline{c}]$. Similarly it can be shown that

$$U_b(v) - U_b(\underline{v}) = \int_{\underline{v}}^{v} Q_b(x) dx \tag{16}$$

Rearranging (15) and (16) shows that in any Nash equilibrium of any bargaining game

$$U_s(c) = U_s(\overline{c}) - \int_c^{\overline{c}} (1 - Q_s(x))\, dx \tag{17}$$

and

$$U_b(v) = U_b(\underline{v}) + \int_{\underline{v}}^{v} Q_b(x) dx \tag{18}$$

2. Next we show that (ϕ_s, ϕ_b) satisfies individual rationality iff $U_s(\overline{c}) \geq \overline{c}$ and $U_b(\underline{v}) \geq 0$. Necessity is clear. Assume that $U_s(\overline{c}) \geq \overline{c}$. Since $Q_s(c) \geq 0$ for all c from (17) we can conclude that $U_s(c) \geq U_s(\overline{c}) - \int_c^{\overline{c}} 1 dx = U_s(\overline{c}) - \overline{c} + c \geq c$. The buyer's case goes similarly.

3. Next we substitute $U_s(c) = (1 - Q_s(c))\, c + T_s(c)$ and $U_b(v) = Q_b(v)v - T_b(v)$ into (17) and (18) and get the following expressions

$$T_s(c) = U_s(\overline{c}) - (1 - Q_s(c))\, c - \int_c^{\overline{c}} (1 - Q_s(x))\, dx \tag{19}$$

$$T_b(v) = -U_b(\underline{v}) + Q_b(v)v - \int_{\underline{v}}^{v} Q_b(x) dx \tag{20}$$

Assume that the Nash equilibrium is ex post efficient. It must be the case that

$$Q_s(c) = \Pr_v [c < v \,|c] \tag{21}$$

and

$$Q_b(v) = \Pr_c [c < v \,|v] \tag{22}$$

This means, by (19) and (20), that $T_s(c)$ and $T_b(v)$ and also $E_{c,v} [T_b(v) - T_s(c)]$ are determined by $U_s(\overline{c})$ and $U_b(\underline{v})$. This gives us the following result: Consider any two bargaining games and any two Nash equilibria of these games. Assume that the Nash equilibria are ex post efficient. If $U_s(\overline{c})$ is the same in both Nash equilibria then also T_s is the same in both Nash equilibria. If $U_b(\underline{v})$ are the same in both Nash equilibria then also T_b is the same in both Nash equilibria.

4. Let us find a bargaining game that is ex post efficient, individually rational, and that makes $E_{c,v} [T_b(v) - T_s(c)]$ as large as possible. This means that $U_s(\overline{c})$ and $U_b(\underline{v})$ should be as small as possible. But we know that the smallest possible values are $U_s(\overline{c}) = \overline{c}$ and $U_b(\underline{v}) = 0$.

It could be that there does not exist a Nash equilibrium that satisfies the above conditions, but if it exists it maximises $E_{c,v} [T_b(v) - T_s(c)]$ over all ex post efficient, individually rational Nash equilibria in all bargaining games.

Let us consider the following particular bargaining game. The seller announces $c^* \in [\underline{c}, \overline{c}]$ and the buyer announces $v^* \in [\underline{v}, \overline{v}]$. Trade takes place iff $v^* \geq c^*$. In this case the buyer pays $\max\{c^*, \underline{v}\}$ and the seller receives $\min\{v^*, \overline{c}\}$. If there is no trade there are no payments. It is a weakly dominant strategy to announce truthfully, i.e. $\phi_s(c) = c$ and $\phi_b(v) = v$. This equilibrium is ex post efficient. Moreover, $U_s(\overline{c}) = \overline{c}$ since if $c = \overline{c}$ then either $v < \overline{c}$, in which case there is no trade, or $v \geq \overline{c}$ in which case trade takes place and the seller receives $\overline{c}$. Thus, in all cases the seller gets payoff $\overline{c}$. Similarly, it can be shown that $U_b(\underline{v}) = 0$.

The equilibrium of this game maximises $E_{c,v}\left[T_b(v) - T_s(c)\right]$ over all ex post efficient, individually rational Nash equilibria of all bargaining games. But in this equilibrium for any (c, v) such that $v > c$

$$t_b(c, v) - t_s(c, v) = c - v < 0 \tag{23}$$

This implies that $E_{c,v}\left[T_b(v) - T_s(c)\right] < 0$. In words (23) says that whenever trade takes place a third party must subsidise the players by an amount equal to the total surplus from trade, namely $v - c$. ■